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NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS

TECHNICAL MEMORANDUM 1255

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By F. R. Gantmacher and L. M. Levin

Translation

*"Ob Uravneniakh Dvizhenia Rakety."
Prikladnaya Matematika i Mekhanika, Vol. XI, No. 3, 1947

Washington
April 1950

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EQUATIONS OF MOTION OF A ROCKET*

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The equations of motion of a rocket are given in general form; a rocket is defined as an apparatus with a liquid or powder rocket motor.

1. THEOREM OF MOMENTUM AND THEOREM OF MOMENTS

In considering the motion of the rocket, at each instant of time only the state of those material particles which at that instant are within the control surface passing through the exterior surface of the body of the rocket and the exit section of the nozzle shall be included.

In order to obtain the equations of motion of the rocket, the following procedure is used. An arbitrary but fixed instant of time is considered. A fictitious solid body is denoted by S with mass m, which would be obtained if the rocket at the instant t solidified and ceased giving off particles. The solid body S will not be homogeneous; in some of its parts, it will have the density of a metal and in other parts the density of a gas, and so forth. It shall be assumed that the fictitious solid body S is invariably fixed to the body of the rocket and from the instant t onwards (instant of solidification) moves together with the rocket. The momentum of the body S shall be denoted by Q.

The system Σ consisting of all the material particles that at the instant t entered the composition of the rocket shall also be considered. At the instant t the system Σ coincides with the rocket, but at the succeeding instants certain of the particles of the system Σ will be outside the rocket. The system Σ and the solid body S have a constant mass equal to the mass of the rocket at the instant of time t. The momentum of the system Σ relative

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*"Ob Uravneniakh Dvizhenia Rakety." Prikladnaya Matematika i Mekhanika, Vol. XI, No. 3, 1947, pp. 301 - 312.

to an immovable (more accurately Galilean) system of coordinate axes will be denoted by K. Let $F = dK/dt$ be the principal vector of the external forces acting at the instant t on the rocket (and therefore on the system Σ).

Compare dK/dt and dQ/dt. The motion of each of the particles of the system Σ shall be considered as compounded. A particle moves relative to S (that is, relative to the body of the rocket) but the solid body S is transported as a whole. For the absolute relative and transport velocities, the notation v_a, v_r, and v_e shall be used. Similarly for the accelerations, the notation w_a, w_r, and w_e is used. The Coriolis acceleration of the particle shall be denoted by J.

It is noted that v_r is equal to zero for the particles of the body[1] and for the particles of the powder, whereas w_r is equal to zero for the particles of the body and those particles of the powder, which at the given instant, do not lie on the combustion surface. Then[2]

$$\frac{dK}{dt} = \sum mw_a = \frac{dQ}{dt} - J + \sum mw_r \tag{1.1}$$

where $J = \sum mj$ is the principal vector of the Coriolis forces and

$$\frac{dQ}{dt} = \sum mw_e \tag{a}$$

The relative velocity of the particle at the instant $t_1 = t + dt$ is denoted by v_{1r}. For the elementary change in

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[1]In the body are included all fixed particles of the rocket.

[2]In the following discussion, the subscript i to denote magnitudes relative to the particles will, for simplification, be ommitted; for example, v_{ai} and m_i will be written as v_a and m, respectively.

velocity δv_r (relative to the body of the rocket)[3], $\delta v_r = v_{1r} - v_r = w_r dt$, hence

$$\sum m w_r dt = \sum m v_{1r} - \sum m v_r \tag{1.2}$$

The momentum relative to the body of the rocket at the instant t of the particles of the gas at that instant in the rocket is denoted by K^r and the momentum relative to the body of the rocket of the particles of the gas[4] in the rocket at instant t by $K_1{}^r$. Now

$$\left.\begin{aligned} \delta K^r &= K_1{}^r - K^r \\ \sum m v_r &= K^r \\ \sum m v_{1r} &= K_1{}^r + k_r dt \end{aligned}\right\} \tag{1.3}$$

where $k_r dt$ is the momentum (in the relative motion) of those particles of the gas, which in the time interval dt passed through the exit section of the nozzle; k_r is the momentum relative to the body of the rocket of the mass of gas passing per second through the section of the nozzle or, as used herein, the momentum rate per second relative to the body of the rocket; δK^r is the elementary change of the relative momentum of the gas occupying a fixed volume (within the control surface).

The vector k_r has the dimensions of a force.

The force k_r is called the equilibrant of the reaction forces or simply the reaction force.[5] If $\mu = - dm/dt$ is the mass flow

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[3]Here and in the following discussions, the symbol δ denotes the differential (elementary change) of the vector relative to the body of the rocket. The elementary change relative to the initial system of axes (fixed) is denoted by d.

[4]For a rocket with liquid reaction engine, the vector K^r is the momentum of the particles of gas in the combustion chamber and nozzle and the particles of the liquid moving in the tanks and the pipes supplying fuel to the combustion chamber.

[5]In the equivalent reactive force there are often included certain external forces; these will be considered more in detail in section 2.

per second and u (the average at the exit section), the velocity of flow of the gas relative to the nozzle[6] $k_r = \mu u$. From equations (1.2) and (1.3), there is obtained

$$\sum m w_r = k_r + \frac{\delta K^r}{dt} \tag{1.4}$$

Substitution on the right side of equation (1.1) for $\sum m w_r$ of the expression from equation (1.4) and F for dK/dt yields

$$\frac{dQ}{dt} = F - k_r + J - \frac{\delta K^r}{dr} \tag{1.5}$$

This equation expresses the momentum theorem for the solidified rocket, that is, for the solid body S.

The kinetic moments of the system Σ and the body S are now considered.

The notation Λ, the kinetic moment of the system Σ, and L of the body S in the absolute motion, that is, in the motion relative to the fixed system of the coordinate axes, are introduced. The pole relative to which the kinetic moment is taken is denoted by a subscript; thus, for example, L_c is the kinetic moment of the body S relative to its center of inertia C and Λ_{c1} is the kinetic moment of the system relative to its center of inertia C_1.

Together with the absolute motion, the motion relative to axes passing through the point C and moving forward together with it must be considered.[7] The magnitudes referring to this motion shall be denoted by a prime.

Similarly in considering the motion relative to axes passing through C_1 (the center of inertia of the system Σ) and having a forward motion the corresponding magnitudes will be denoted by a double prime.

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[6] If the rocket has several nozzles, $k_r = \sum \mu_i u_i$, where μ_i is the gas flow of the i th nozzle and u_i is the mean velocity at the exit section of this nozzle. In the following discussion, a rocket with a single nozzle is considered; this assumption does not affect the generality of the results.

[7] The origin of this system of coordinate axes C is not displaced relative to the body of the rocket.

Applying the theorem of the moment of momentum relative to the center of inertia to the system Σ, there is obtained for the instant of time t

$$\frac{d\Lambda_{c1}''}{dt} = G_{c1} \tag{1.6}$$

where G_{c1} is the principal moment of all the external forces acting on the rocket (and therefore on the system Σ) at the instant of time t.

It is noted that $\Lambda_c'' = \Lambda_c'$. For, in passing to another system of coordinate axes moving translationally relative to the first, there is added to the velocities of all points of the system the same velocity constant in magnitude and direction. The additional momentums will be proportional to the masses and are in the same directions. Hence, they reduce to a single equivalent resultant vector applied to the center of inertia C_1. The moment of this additional momentum vector relative to C_1 will be equal to zero. Further

$$\Lambda_{c1}' = \Lambda_c' + \overline{C_1C} + K'$$

whence

$$\frac{d\Lambda_{c1}'}{dt} = \frac{d\Lambda_c'}{dt} + \frac{\overline{dC_1C}}{dt} \times K' + \overline{C_1C} \times \frac{dK'}{dt} = \frac{d\Lambda_c'}{dt} \tag{1.7}$$

because at the instant t the point C_1 coincides with the point C and

$$\frac{\overline{dC_1C}}{dt} = v_c - v_{c1} = -\frac{1}{m} K'$$

By noting that at the instant t the points C_1 and C coincide and therefore $G_{c1} = G_c$ and recalling that $\Lambda_{c1}'' = \Lambda_{c1}'$ there is obtained from equations (1.6) and (1.7)

$$\frac{d\Lambda_c'}{dt} = G_c \tag{1.8}$$

Again the motion of each particle of the system Σ shall be considered as compounded. The motion of the particle relative to the axes moving forward together with the point C shall be considered as absolute, the motion of the body S relative to these axes as the transport motion, and finally the motion of the particle relative to the body S (that is, relative to the body of the rocket) as relative. Then

$$\Lambda_c' = \sum r \times m v_a$$
$$\frac{d\Lambda_c'}{dt} = \sum r \times m w_a$$

where r is the radius vector of the particle drawn from the point C. The sum $w_r + w_e + j$ is substituted instead of the absolute acceleration of the particle and it is noted that

$$\left.\begin{aligned} \sum r \times m w_e &= \frac{dL_c'}{dt} \\ -\sum r \times m j &= H_c \end{aligned}\right\} \qquad (1.9)$$

where H_c is the principal moment of the Coriolis forces. There is obtained

$$\frac{d\Lambda_c'}{dt} = \frac{dL_c'}{dt} - H_c + \sum r \times m w_r \qquad (1.10)$$

Again let v_{1r} be the relative velocity of the particle at the instant $t_1 = t + dt$. Then $v_{1r} - v_r = w_r\,dt$ and therefore

$$\sum r \times m w_r dt = \sum r \times m v_{1r} - \sum r \times m v_r \qquad (1.11)$$

It is noted that

$$\sum r \times m v_r = \Lambda_c^r \qquad (1.12)$$

where Λ_c^r is the kinetic moment of the gas within the rocket in the relative motion[8] at the instant t. The value of this kinetic moment of the gas at the instant of time t_1 is denoted by Λ_{1c}^r. Further

$$\sum r \times m v_{1r} = \sum r_1 \times m v_{1r} - \sum (r_1 - r) \times v_{1r}$$

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[8]See the previous note relative to K^r. The magnitudes Λ_c^r like K^r refers not to the fixed mass but to a fixed volume occupied by the gas.

where r_1 is the radius vector at the instant t_1 of the particle, which at the instant t had the radius vector r.

From the sum $\sum r_1 \times mv_{1r}$, those components in which the end of the radius vector r_1 extends beyond the limits of the rocket are taken out. The sum of these components will be equal to the kinetic moment per second of the gas through the exit section of the nozzle in the relative motion l_{rc} multiplied by dt.

The sum of the remaining components gives Λ_{1c}^{r}

Further the sum $\sum (r_1 - r) \times mv_{1r} = 0$, because $r_1 - r = \delta r = v_r\, dt$ and with an accuracy up to infinitesimals $v_{1r} \approx v_r$. Thus

$$\sum r \times mv_{1r} = \Lambda_{1c}^{r} + l_{rc} \tag{1.13}$$

From equations (1.11), (1.12), and (1.13),

$$\sum r \times mw_r = \frac{\delta \Lambda_c^r}{dt} + l_{rc} \tag{1.14}$$

where $\delta\Lambda_c^{r} = \Lambda_{1c}^{r} - \Lambda_c^{r}$ is the elementary change relative to the body.

By substituting in equation (1.10) in place of the sum its expression from equation (1.14), there is obtained

$$\frac{d\Lambda_c'}{dr} = \frac{dL_c'}{dt} + l_{rc} - H_c + \frac{\delta \Lambda_c r}{dt} \tag{1.15}$$

The kinetic moment of the gas per second in the relative motion l_{rc} has the dimensions of the moment of a force. The moment l_{rc} is called the reactive moment.[9] From equations (1.8) and (1.15), there is obtained

$$\frac{dL_c'}{dt} = G - l_{rc} + H_c - \frac{\delta \Lambda_c r}{dt} \tag{1.16}$$

Equation (1.16) determines the derivative with respect to time of the kinetic moment of the solidified rocket S in its motion

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[9]In the reactive moment, there are often included additional moments of certain external forces. See section 2.

relative to axes passing through the center of inertia C of the forward moving body S.

2. REACTIVE FORCES

The area of the exit section of the nozzle is divided into elementary areas $d\sigma$. The mass per second flowing through the area $d\sigma$ is denoted by $\nu d\sigma$ and the relative velocity of the gas passing through this area by v_r. The mass of gas per second passing through the exit section of the nozzle will then be $\mu = \sum \nu d\sigma$, where the summation is taken over all the elements $d\sigma$ of the exit section.

The vector $- \nu d\sigma v_r$, having the dimensions of a force, are considered. This force is called the elementary reactive force. It originates in the particles of the gas separating from the rocket through the area $d\sigma$. The principal vector of the elementary reactive forces is equal to $- k_r$, and the principal moment relative to the pole C is equal to $- l_{rc}$ where k_r and l_{rc} are the momentums per second and kinetic moment of the gas relative to the body of the rocket.

In the system of reactive forces there are, however, often included certain external forces, namely, those arising from the atmospheric pressure on the body of the rocket and from the pressure of the issuing parts of the gas on those remaining in the rocket, and additional forces due to the unsteady motion of the gas. This phenomenon is made clear by the following considerations.

The combustion in the rocket apparatus is assumed to be at standard conditions with the rocket immovably fixed. In this case

$$Q = 0 \qquad H_c = 0$$

$$L_c' = 0 \qquad F = F_* + F^*$$

$$J = 0 \qquad G_c = G_{c*} + G_c^*$$

where F_* and G_{*c} are the principal vector and the principal moment of the forces of the atmospheric pressure and pressure of the external part of the gas on the exit section of the nozzle.

From equations (1.5) and (1.6), then[10]

$$- F_* = - k_r + F^* - \frac{\delta K^r}{dt}$$

$$- G_{*c} = - l_{rc} + G_c^* - \frac{\delta \Lambda_c^r}{dt}$$

In measurements on the test stand there are generally determined the pressure forces of the rocket on the supports. These pressures are characterized by the principal vector $-F$ and the principal moment $-G_{*c}$, which include in addition to the purely reactive forces the additional forces and moments F^*, G_c^*, $-\delta K^r/dt$, $-\delta \Lambda^r{}_c/dt$.

In the following discussion into one system shall be combined: (1) the purely reactive forces, (2) the forces arising from the atmospheric pressure[11] and the pressure of the external part of the gas (issuing from the rocket jet), and (3) the additional forces due to the nonsteadiness of the motion of the gas in the rocket. All these forces will be included in the system of reactive forces. The principal vector of these forces $-F_*$ is denoted by T and the principal moment $-G_{*c}$ by M_c. Then

$$\left.\begin{aligned} T &= - k_r + F^* - \frac{\delta K^r}{dt} \\ M_c &= - l_{rc} + G_c^* - \frac{\delta \Lambda^r{}_c}{dt} \end{aligned}\right\} \qquad (2.1)$$

Generally in computing the reactive force T and the reactive moment M_c the third components in equations (2.1), that is, $\delta K^r/dt$ and $\delta \Lambda^r{}_c/dt$ are neglected.[12]

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[10]It is here assumed that the flow of the gas in the chamber and nozzle for the moving and stationary rocket is the same. This assumption is equivalent to neglecting the effect of the acceleration of the rocket on the relative motion of the gas.

[11]By the forces of the atmospheric pressure is meant the forces due to constant atmospheric pressure on the external surface of the rocket at standard conditions.

[12]That is, the motion of the gas in the rocket is considered as quasi-stationary.

3. FINAL FORMULATION OF THE FUNDAMENTAL THEOREMS. PRINCIPLE OF SOLIDIFICATION

From the number of external forces were separated the forces arising from the uniform atmospheric pressure and the pressure of the external gas, which were included (section 2) in the system of reactive forces. These forces have the principal vector F^* and the principal moment G_c^*. The principal vector and the principal moment of the remaining external forces are now denoted by F and G.

With this notation, the equation of the change of momentum (1.5) and the equation of the kinetic moment for the solidified rocket (1.16) assume the form

$$\frac{dQ}{dt} = F + F^* - k_r + J - \frac{\delta K^r}{dt}$$

$$\frac{dL_c'}{dt} = G_c + G_c^* - l_{cr} + H_c - \frac{\delta \Lambda_c^{\ r}}{dt}$$

By introducing these equations the reactive force T and the moment M_c according to equations (2.1), there is finally obtained

$$\left.\begin{aligned} \frac{dQ}{dt} &= F + T + J \\ \frac{dL_c'}{dt} &= G_c + M_c + H_c \end{aligned}\right\} \qquad (3.1)$$

From these equations the following principle of solidification is obtained:

The equations of motion of a rocket at an arbitrary instant of time t may be written in the form of the equations of motion of a solid body of constant mass if it is assumed that the rocket became rigid and solidified at the instant of time t (that is, ceased to give out particles) and that to the fictitious solid body thus obtained there were applied: (1) external forces acting on the rocket, (2) reactive forces, and (3) Coriolis forces.

4. SECOND DERIVATION OF THE EQUATION OF MOMENTUM AND THE EQUATION OF MOMENTS

An infinitely small interval of time dt is considered and the elementary increments, for this interval, of dK and dQ the momentum of the system Σ, and the solid body S are compared.

For this purpose, the volume occupied by the rocket is divided into three parts: (1) the volume occupied by the body, (2) the volume occupied at the instant t by the powder (or liquid) fuel, and (3) the volume occupied at the instant t by the particles of gas in the rocket. The parts of the momentum referring to these volumes are denoted by the subscripts 1, 2, and 3, respectively.

Evidently for the body,

$$dK_1 = dQ_1 \tag{4.1}$$

The powder is next considered. At the instant t, $K_2 = Q_2$; but at the instant $t_1 = t + dt$ the vector K_2 differs from Q_2 by the momentum of the elementary mass of powder that burned in the time interval dt. This deficiency of momentum is presented in the form $k_p dt$ where k_p is the momentum expenditure per second of the powder. Then[13]

$$dK_2 = dQ_2 - k_p dt \tag{4.2}$$

The volume of the part of the rocket occupied at the instant t by the gas is now considered. The momentum of the particles of gas

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[13]The momentum of the gas occupying at the instant t_1 the volume of burned powder after dt seconds is not considered here. The ratio of this momentum to the value $k_p dt$ is equal to the ratio of the density of the gas to the density of the powder and is very small. Thus by k_p, the expenditure per second of the momentum of the powder multiplied by $1 - \epsilon$ is known. For the rocket with liquid fuel $dK_2 = dQ_2 - k_p dt + dK_0{}^r$ where $K_0{}^r$ is the momentum of the liquid fuel relative to the motion and k_p the consumption per second of the momentum of the fuel in the transport motion multiplied by $1 - \epsilon$, where ϵ is the ratio of the density of the fuel vapors to the density of the fuel itself.

in this volume relative to the body of the rocket is denoted (as in section 1) by K^r. The corresponding increment is dK^r, where the differential d takes account of the vector relative to the stationary system of coordinates. But then

$$dK^r = \delta K^r + \boldsymbol{\omega} \times K^r \, dt$$

where δK^r is the elementary change of the vector K^r relative to the body of the rocket and $\boldsymbol{\omega}$ is the angular velocity of the body.

The increment of momentum of the gas in the volume considered in the transport motion is now considered. This increment agrees with dQ_3 in the case where at each point of the volume considered the density of the gas ρ at the instants t and t_1 is the same. In the general case, however, it is necessary to take into account also the change in density ρ. Hence, in the general case the required increment in the transport motion is equal to

$$dQ_3 + \iiint \frac{\partial \rho}{\partial t} \, dt \, \mathbf{v}_e d\boldsymbol{\tau}$$

where the integral is extended over that part of the volume W that at the instant t is occupied by the gas, v_e is the transport velocity of the element of volume $d\boldsymbol{\tau}$, and $\partial\rho/\partial t$ is the rate of change of the density ρ in the given volume element $d\boldsymbol{\tau}$. Hence

$$dK_3 = dQ_3 + \iiint \frac{\partial \rho}{\partial t} \, dt v_e d\boldsymbol{\tau} + \delta K^r + \boldsymbol{\omega} \times K^r \, dt \qquad (4.3)$$

The increments in the momentum of the particles within the volume of the rocket have been considered. In computing dK, however, it is necessary to take into account also the moment at the time t_1 of those particles, which in the interval from t to $t_1 = t + dt$ passed through the exit section of the nozzle. This momentum is equal to $(k_e + k_r)dt$ where k_e is the rate of expenditure per second of the momentum of the gas through the exit section of the nozzle in the transport motion and k_r in the relative motion. Thus

$$dK = dK_1 + dK_2 + dK_3 + (k_e + k_r)dt$$

Substituting dK_1, dK_2, and dK_3 from equations (4.1), (4.2) and (4.3), respectively, there is obtained

$$dK = dQ - (k_p - k_e - k_r)dt + \delta K^r + \omega \times K^r\, dt + \iiint \frac{\partial \rho}{\partial t}\, dt\, v_e\, d\tau \qquad (4.4)$$

As before, the principal vector of the forces of the atmospheric pressure and the pressure of the external part of the gas on the rocket is denoted by F^* and the principal vector of the remaining external forces by F. According to the momentum theorem for the system Σ , $dK/dt = F + F^*$ and therefore from equation (4.4) there is obtained [14]

$$\frac{dQ}{dt} = F - k_r + F^* - \frac{\delta K^r}{dt} + k_p - k_e - \iiint \frac{\partial \rho}{\partial t}\, v_e\, d\tau - \omega \times K^r \qquad (4.5)$$

where the last term $- \times K^r$ represents half the Coriolis force J. Hence, by taking equations (2.1) into account, equation (4.5) can be reduced to the form

$$\frac{dQ}{dt} = F + T + k_p - k_e - \iiint \frac{\partial \rho}{\partial t}\, v_e\, d\tau + \frac{1}{2} J \qquad (4.6)$$

Similar considerations permit obtaining the equation of moments

$$\frac{dL'_c}{dt} = G_c + M_c + l_{cp}' - \omega \times \Lambda_c^r - \iiint r' \times v_e' \frac{\partial \rho}{\partial t}\, d\tau \qquad (4.7)$$

where l_{cp} is the kinetic moment of the powder per second and l_{ce} is the kinetic moment of the gas per second through the exit section of the nozzle. In both cases, there is considered the kinetic moment in the transport motion of the body relative to a system of axes moving forward together with the center of inertia C of the body S.

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[14]This equation holds also for the liquid fuel rocket. In this case, however, K^r is the momentum of the gas and the fuel in their relative motion.

5. EQUATION FOR THE MOMENTUM, CORIOLIS FORCE, AND CORIOLIS MOMENT

By comparing equations (4.6) and (4.7) with equations (3.1), the equations for the Coriolis force J and the Coriolis moment H_c are obtained:

$$J = 2\left(k_p - k_e - \iiint \frac{\partial\rho}{\partial t} v_e \, d\tau\right) \tag{5.1}$$

$$H_c = l_{cp}' - l'_{ec} - \omega \times \Lambda_c{}^r - \iiint \frac{\partial\rho}{\partial t} r' \times v_e' \, d\tau \tag{5.2}$$

These equations are evidently simplified if the density of the gas does not vary with time, that is, $\partial\rho/\partial t = 0$. (This equation holds for the quasi-stationary motion of the gas in the rocket.)

The expressions (5.1) for J are first considered. The mass of powder expended per second (or what amounts to the same thing, the addition of the mass of gas) is denoted by μ_1 and the mass of gas per second through the exit section of the nozzle by μ_2. Then evidently

$$\mu_1 - \mu_2 = \frac{d}{dt}\iiint \rho \, d\tau = \iiint \frac{\partial\rho}{\partial t} \, d\tau \tag{5.3}$$

Further, $k_p dt$ represents the momentum of an elementary mass of powder $\mu_1 dt$ burning in the interval of time dt. The center of inertia of this elementary mass of powder is denoted by C_p and the velocity of the point C_p in the transport motion of the body by v_{cp}. Then $k_p = \mu_1 v_{cp}$.

If it is assumed that the elementary mass of powder $\mu_1 dt$ is symmetrically placed relative to the axis of the rocket, the point C_p will lie on this axis. If the mass $\mu_1 dt$ is symmetrical with respect to the mean section of the powder containers, C_p will be in the plane of this section.

In the same manner, the elementary mass of gas $\mu_2 dt$ passing through the exit section of the nozzle in time dt is considered. By C_e the center of inertia of this elementary mass is denoted and by v_{ce} the transport velocity of this center. Then $k_e = \mu_2 v_{ce}$.

If the gas passing through the exit section is symmetrical, the point C_e lies on the center of the exit section of the nozzle.

By r_p and r_e the radii vectors of the points C_p and C_e, respectively, drawn from any pole 0 fixed to the body are denoted.

Let r denote the radius vector of an arbitrary point within the rocket. By using the expression for the velocities of the points of a solid body,

$$\left.\begin{aligned} v_{cp} &= v_0 + \boldsymbol{\omega} \times r_p \\ v_{c0} &= v_0 + \boldsymbol{\omega} \times r_e \\ v_e &= v_0 + \boldsymbol{\omega} \times r \end{aligned}\right\} \tag{5.4}$$

On the basis of these equations and the equations $k_p = \mu_1 v_{cp}$ and $k_e = \mu_2 v_{ce}$, equation (5.1), taking account of equation (5.3), assumes the form

$$J = -2\boldsymbol{\omega} \times \left(\mu_2 r_e - \mu_1 r_p + \iiint \frac{\partial \rho}{\partial t} r \, d\tau\right) \tag{5.5}$$

By comparing this equation with the usual expression for the Coriolis force $J = -2\boldsymbol{\omega} \times K^r$ and taking into account the arbitrariness of the vector $\boldsymbol{\omega}$, there results

$$K^r = \mu_2 r_e - \mu_1 r_p + \iiint \frac{\partial \rho}{\partial t} r \, dt \tag{5.6}$$

where K^r is the momentum of the gas in the rocket in its motion relative to the rocket and the integral on the right side is extended over the entire volume W occupied by the gas.

Equation (5.6) is of a general character.[15] It determines the momentum of the gas (or liquid) enclosed in the given volume in the

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[15]Equations (5.6) and (5.2) for K^r and H_c can be obtained directly if the derivatives are computed with respect to time of the integrals

$$\iiint r dm \qquad \iiint r \times (\boldsymbol{\omega} \times r) \, dm$$

taken over the mass occupying the given volume at the time t.

case where the incoming and outgoing masses μ_1 and μ_2 of the gas through the surface bounding this volume and the per second change in density at each point of the volume are given. In this equation, r_p is the radius vector of the center of inertia of the incoming mass per second μ_1 and r_e the radius vector of the outgoing mass μ_2.

The particular case where $\partial\rho/\partial t \approx 0$ (for example the quasi-stationary motion of a gas, the arbitrary motion of an incompressible liquid[16], and so forth) is considered. In this case according to equation (5.3), there is $\mu_1 = \mu_2 = \mu$ where for the case of the rocket $\mu = dm/dt$. If the vector $b = \overline{C_eC_p}$ is introduced, equation (5.6) and the equation for J become

$$K^r = \mu b \qquad J = -2\boldsymbol{\omega} \times K^r = -2\mu(\boldsymbol{\omega} \times b) \qquad (5.7)$$

Equation (5.2) for the Coriolis moment H_c is now considered. If it is assumed that $\partial\rho/\partial t = 0$ and that the equivalent vector of the momentum of the particles of gas in their relative motion passes through the center of inertia C, that is, $\boldsymbol{\Lambda}_c^r = 0$, equation (5.2) becomes

$$H_c = l'_{pc} - l'_{ec}$$

For example, the case of plane-parallel motion of the rocket is considered. The moment of inertia of the body relative to the equatorial axis passing through the point C and perpendicular to the plane of the motion is denoted by I. The moment of inertia of the mass of powder burned in dt seconds will then be equal to $-dI$. Hence the kinetic moment of this mass will be

$$l'_{pc}\,dt = -\boldsymbol{\omega}dI$$

On the other hand, the kinetic moment (in the translational motion) of an elementary mass of the gas passing through the exit section in time dt will be $-dm\, r_e^2\boldsymbol{\omega}$ where $r_e = \overline{CC_e}$. Thus

$$H_c = \left(-\frac{dI}{dt} + \frac{dm}{dt}r_e^2\right)\boldsymbol{\omega}$$

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[16] In this case there is the exact equation.

6. EQUATION OF MOTION OF THE CENTER OF INERTIA OF THE ROCKET

From the principle of solidification for the instant of time t, there results

$$m w_c = F + T + J \tag{6.1}$$

where m is the mass of the rocket at instant t and w_c is the acceleration of the center of inertia C of the solidified rocket at time t, that is, of the body S.

During the combustion of the fuel, however, the center of inertia of the rocket is displaced relative to the body. The motion of the center of inertia of the rocket relative to the initial (fixed) system of coordinates can be represented in the form of a compound motion in which the center of inertia moves relative to the body (relative motion) and the body of the rocket moves relative to the fixed system of axes (transport motion). Then v_c and w_c are the transport velocity and acceleration and v_{cr} and w_{cr} the relative values. The absolute velocity and the acceleration of the center of inertia are determined from the equations

$$v = v_c + v_{cr}$$

$$w = w_c + w_{cr} + 2\omega \times v_{cr}$$

Determining from the second equation w_c and substituting in equation (6.1), there is obtained the equation of motion of the center of inertia of the rocket.

$$m w = F + T + J + m w_{cr} + 2m\omega \times v_{cr} \tag{6.2}$$

The expressions for the magnitudes v_{cr} and w_{cr} shall be found. The body of the rocket may here be considered as fixed. Let the point O of the body of the rocket be taken as the initial and C and C' the position of the center of inertia of the rocket at the instant t and $t' = t + dt$. Then for the moments of the times t and t', there results

$$m r_c = \iiint \rho\,(t)\, r\, d\tau$$

$$(m + dm)\, r_c' = \iiint \rho\,(t + dt)\, r\, d\tau$$

where $r_c = \overline{OC}$, $r_{c'} = \overline{OC'} = r_c + \delta r_c$, and ρ is the density of the elementary volume of the rocket. The integration is extended over the entire volume of the rocket (including the body).

By subtracting the first relation from the second and neglecting the terms of the second-order smallness,

$$dm\, r_c + m\, \delta r_c = -\mu_1 r_p + \iiint \frac{\partial \rho}{\partial t} dt\, r\, d\tau$$

whence it follows that

$$mv_{cr} = m \frac{\delta r_c}{dt} = \mu_2 r_c - \mu_1 r_p + \iiint \frac{\partial \rho}{\partial t} r\, d\tau$$

or on the basis of equation (5.6)

$$mv_{cr} = \mu_2 (r_c - r_e) + K^r \tag{6.3}$$

By differentiating the equation (6.3),

$$-\mu_2 v_{cr} + mw_{cr} = \mu_2 v_{cr} + \frac{\partial \mu_2}{\partial t} (r_c - r_e) + \frac{\delta K^r}{dt}$$

or

$$mw_{cr} = 2\mu_2 v_{cr} + \frac{\partial \mu_2}{\partial t} (r_c - r_e) + \frac{\delta K^r}{dt} \tag{6.4}$$

In the quasi-stationary case $\partial \rho / \partial t \approx 0$, $\mu_2 = \mu_1 = \mu$, $\delta K^r / dt \approx 0$, $\partial \mu_2 / \partial t \approx 0$, and equations (6.3) and (6.4) assume the form

$$\left.\begin{aligned} v_{cr} &= \frac{\mu}{m} (r_c - r_p) = \frac{\mu}{m} \overline{C_p C} \\ w_{cr} &= 2 \frac{\mu^2}{m^2} (r_c - r_p) = 2 \frac{\mu^2}{m^2} \overline{C_p C} \end{aligned}\right\} \tag{6.5}$$

The computations conducted by these equations show that the magnitudes v_{cr} and w_{cr} are negligibly small in comparison with the mean velocities and the accelerations on the active part of the trajectory. Hence in equation (6.2), the term mw_{cr} may be neglected in comparison with the reactive thrust T. The term $2m\omega \times v_{cr}$ is generally of the same order as J because $|r_c - r_p|$ is of the same order as $|r_e - r_p|$.

7. EQUATIONS OF THE ROTATIONAL MOTION OF THE ROCKET

By ξ, η, and ζ are denoted principal central axes of inertia of the rocket, by I_1, I_2, and I_3 the moments of inertia of the rocket relative to these axes, by p, q, and r the projections of the angular velocity of the body ω on these axes.

It is first assumed that the directions of the axes of inertia ξ, η, and ζ are fixed relative to the body during the entire time of combustion. This natural assumption is always made for rockets because it is assumed that at all times the axis of the rocket is one of the principal axes of inertia and the two other principal axes of inertia may be arbitrarily chosen in the plane perpendicular to the axis of the rocket.

Under the assumption made, the principal axes of inertia of the solidified rocket, that is, of the body S, will at all times be parallel to the axes ξ, η, and ζ. The corresponding moments of inertia for the body S will have constant values equal to the values of the moments of inertia I_1, I_2, and I_3 of the rocket at the instant of time t. It follows that p, q, and r are the projections of the angular velocities of the solid body S on the principal central axes of inertia of this body. Hence, making use of the principle of solidification, the three equations of the rotational motion can be written in the form of Euler[17]

$$I_1 \frac{dp}{dt} + (I_3 - I_2)\,qr = N_\xi \qquad (\xi,\ \eta,\ \zeta;\ 1,\ 2,\ 3;\ p,\ q,\ r) \tag{7.1}$$

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[17]Here and in the following discussion, the parentheses after each equation indicate that two other equations are obtained by cyclical interchange of the letters and subscripts in the parentheses.

where N_ξ is the sum of the moments relative to the axis ξ of all the external reactive and Coriolis forces and similarly for N_η and N_ζ.

The three equations (7.1) together with the three scalar equations of motion of the center of inertia constitute a system of six equations, which determines the motion of the rocket.

For completeness, the general case when the principal central axes of inertia change their directions relative to the body of the rocket shall be considered.

By ξ', η', and ζ' the directions of the principal axes of inertia of the body S is denoted, that is, the directions fixed relative to the body and coinciding with the directions ξ, η, and ζ at the time t. The projections of the angular velocity ω on the axes ξ', η', and ζ' are denoted by p', q', and r' respectively. Applying the principle of solidification, for the moment of time t

$$I_1 \frac{dp'}{dt} + (I_3 - I_2)\, q'\, r' = N_\xi \qquad (\xi, \eta, \zeta;\ 1, 2, 3;\ p, q, r) \tag{7.2}$$

where $N_\xi{}'$ is the sum of the projection of the external reactive and Coriolis forces on the axis ξ', and so forth. By $\boldsymbol{\Omega}$ the angular velocity of the trihedron $\xi\eta\zeta$ relative to the body of the rocket is denoted. Let $d\omega/dt$ be the derivative of the vector relative to the body of the rocket and $\delta\omega/dt$ the derivative relative to the axes $\xi\eta\zeta$. Then

$$\frac{d\omega}{dt} = \frac{\partial\omega}{dt} + \boldsymbol{\Omega} \times \omega \tag{7.3}$$

The projections of the vector $d\omega/dt$ on the axes ξ', η', and ζ' are equal to dp'/dt, dq'/dt, and dr'/dt. The projections of the vector $\delta\omega/dt$ on the axes ξ, η, and ζ are equal to dp/dt, dq/dt, and dr/dt. Because at the instant t the directions ξ', η', and ζ' coincide with the directions η, and ζ for this instant of time, from equation (7.3)

$$\frac{dp'}{dt} = \frac{dp}{dt} + \boldsymbol{\Omega}_\eta r - \boldsymbol{\Omega}_\xi q \qquad (\xi, \eta, \zeta;\ p, q, r) \tag{7.4}$$

It is noted that $p' = p$, $q' = q$, and $r' = r$ at the instant t. Substituting in equation (7.2) in place of dp'/dt, dq'/dt, and dr'/dt their expressions from equation (7.4) there is finally obtained

$$\delta \qquad \left. \begin{aligned} I_1 \frac{dp}{dt} + (I_3 - I_2)\, qr + I_1 (\Omega_\eta r - \Omega_\zeta q) = N_\xi \\ (\xi, \eta, \zeta;\ 1, 2, 3;\ p, q, r) \end{aligned} \right\} \qquad (7.5)$$

where Ω_ξ, Ω_η, and Ω_ζ must be considered as known functions of the time.

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National Advisory Committee
for Aeronautics.

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